SIGN-GRADED POSETS, UNIMODALITY OF W-POLYNOMIALS AND THE CHARNEY-DAVIS CONJECTURE

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ABSTRACT. We generalize the notion of graded posets to what we call sign-graded (labeled) posets. We prove that the W-polynomial of a sign-graded poset is symmetric and unimodal. This extends a recent result of Reiner and Welker who proved it for graded posets by associating a simplicial polytopal sphere to each graded poset. By proving that the W-polynomials of sign-graded posets has the right sign at -1, we are able to prove the Charney-Davis Conjecture for these spheres (whenever they are flag).

1. INTRODUCTION AND PRELIMINARIES

Recently Reiner and Welker [9] proved that the W-polynomial of a graded poset (partially ordered set) P has unimodal coefficients. They proved this by associating to P a simplicial polytopal sphere, $\Delta_{eq}(P)$, whose h-polynomial is the W-polynomial of P, and invoking the g-theorem for simplicial polytopes (see [13, 14]). Whenever this sphere is flag, i.e., its minimal non-faces all have cardinality two, they noted that the Neggers-Stanley Conjecture implies the Charney-Davis Conjecture for $\Delta_{eq}(P)$. In this paper we give a different proof of the unimodality of W-polynomials of graded posets, and we also prove the Charney-Davis Conjecture for $\Delta_{eq}(P)$ (whenever it is flag). We prove it by studying a family of labeled posets, which we call sign-graded posets, of which the class of graded naturally labeled posets is a sub-class.

In this paper all posets will be finite and non-empty. For undefined terminology on posets we refer the reader to [15]. We denote the cardinality of a poset P with the letter p. Let P be a poset and let $\omega : P \to \{1, 2, \ldots, p\}$ be a bijection. The pair (P, ω) is called a *labeled poset*. If ω is order-preserving then (P, ω) is said to be *naturally labeled*. A (P, ω)-*partition* is a map $\sigma : P \to \{1, 2, 3, \ldots\}$ such that

- σ is order reversing, that is, if $x \leq y$ then $\sigma(x) \geq \sigma(y)$,
- if $x < y$ and $\omega(x) > \omega(y)$ then $\sigma(x) > \sigma(y)$.

The theory of (P, ω)-partitions was developed by Stanley in [12]. The number of (P, ω)-partitions σ with largest part at most n is a polynomial of degree p in n called the *order polynomial* of (P, ω) and is denoted $\Omega(P, \omega; n)$. The W-polynomial of (P, ω) is defined by

$$\sum_{n \geq 0} \Omega(P, \omega; n+1)t^n = \frac{W(P, \omega; t)}{(1-t)^{p+1}}.$$

The set, $\mathcal{L}(P, \omega)$, of permutations $\omega(x_1), \omega(x_2), \ldots, \omega(x_p)$ where $x_1, x_2, \ldots, x_p$ is a linear extension of P is called the *Jordan-Hölder set* of (P, ω). A *descent* in a permutation $\pi = \pi_1 \pi_2 \cdots \pi_p$ is an index $1 \leq i \leq p - 1$ such that $\pi_i > \pi_{i+1}$. The number of descents in π is denoted $\mathrm{des}(\pi)$. A fundamental result on (P, ω)-partitions, see [12], is that the W-polynomial can be written as

$$W(P, \omega; t) = \sum_{\pi \in \mathcal{L}(P, \omega)} t^{\mathrm{des}(\pi)}.$$

The Neggers-Stanley Conjecture is the following:

Conjecture 1.1 (Neggers-Stanley). *For any labeled poset (P, ω) the polynomial $W(P, \omega; t)$ has only real zeros.*

This was first conjectured by Neggers [7] in 1978 for natural labelings and by Stanley in 1986 for arbitrary labelings. The conjecture has been proved for some special cases, see [1, 2, 9, 16] for the state of the art. If a polynomial has only real non-positive zeros then its coefficients form a unimodal sequence. For the W-polynomials of graded posets unimodality was first proved by Gasharov [6] whenever the rank is at most 2, and as mentioned by Reiner and Welker [9] for all graded posets.

For the relevant definitions concerning the topology behind the Charney-Davis Conjecture we refer the reader to [3, 9, 14].

Conjecture 1.2 (Charney-Davis, [3]). *Let Δ be a flag simplicial homology $(d-1)$-sphere, where d is even. Then the h-vector, $h(\Delta, t)$, of Δ satisfies*

$$(-1)^{d/2} h(\Delta, -1) \geq 0.$$

Recall that the nth *Eulerian polynomial*, $A_n(x)$, is the W-polynomial of an anti-chain of n elements. The Eulerian polynomials can be written as

$$A_n(x) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} a_{n,i} x^i (1+x)^{n-1-2i},$$

where $a_{n,i}$ is a non-negative integer for all i. This was proved by Foata and Schützenberger in [5] and combinatorially by Shapiro, Getu and Woan in [10]. From this expansion we see immediately that $A_n(x)$ is symmetric and that the coefficients in the standard basis are unimodal. It also follows that $(-1)^{(n-1)/2} A_n(-1) \geq 0$.

We will in Section 2 define a class of labeled poset whose members we call sign-graded posets. This class includes the class of naturally labeled graded posets. In Section 4 we show that the W-polynomial of a sign-graded poset (P, ω) of rank r can be expanded, just as the Eulerian polynomial, as

$$W(P, \omega; t) = \sum_{i=0}^{\lfloor (p-r-1)/2 \rfloor} a_i(P, \omega) t^i (1+t)^{p-r-1-2i}, \qquad (1.1)$$

where $a_i(P, \omega)$ are non-negative integers. Hence, symmetry and unimodality follow, and $W(P, \omega; t)$ has the right sign at -1. Consequently, whenever the associated sphere $\Delta_{eq}(P)$ of a graded poset P is flag the Charney-Davis Conjecture holds for $\Delta_{eq}(P)$. We also note that all symmetric polynomials with non-positive zeros only, admits an expansion such as (1.1). Hence, that $W(P, \omega; t)$ has such an expansion can be seen as further evidence for the Neggers-Stanley Conjecture.

In [8] the Charney-Davis quantity of a graded naturally labeled poset (P, ω) of rank r was defined to be $(-1)^{(p-1-r)/2} W(P, \omega; -1)$. In Section 5 we give a combinatorial interpretation of the Charney-Davis quantity as counting certain reverse alternating permutations. Finally in Section 7 we characterize sign-graded posets in terms of properties of order polynomials.

2. Sign-graded posets

Recall that a poset P is *graded* if all maximal chains in P have the same length. If P is graded one may associate a *rank function* $\rho : P \to \mathbb{N}$ by letting $\rho(x)$ be the length of any saturated chain from a minimal element to x. The *rank* of a graded poset P is defined as the length of any maximal chain in P. In this section we will generalize the notion of graded posets to labeled posets.

Let (P, ω) be a labeled poset. An element y *covers* x, written $x \prec y$, if $x < y$ and $x < z < y$ for no $z \in P$. Let $E = E(P) = \{(x, y) \in P \times P : x \prec y\}$ be the covering relations of P. We associate a labeling $\epsilon : E \to \{-1, 1\}$ of the covering relations defined by

$$\epsilon(x, y) = \begin{cases} 1 \text{ if } & \omega(x) < \omega(y), \\ -1 \text{ if } & \omega(x) > \omega(y). \end{cases}$$

If two labelings ω and λ of P give rise to the same labeling of $E(P)$ then it is easy to see that the set of (P, ω)-partitions and the set of (P, λ)-partitions are the same. In what follows we will often refer to ϵ as the labeling and write (P, ϵ).

Definition 2.1. Let (P, ω) be a labeled poset and let ϵ be the corresponding labeling of $E(P)$. We say that (P, ω) is *sign-graded*, and that P is ϵ-*graded* (and ω-*graded*) if for every maximal chain $x_0 \prec x_1 \prec \cdots \prec x_n$ the sum

$$\sum_{i=1}^{n} \epsilon(x_{i-1}, x_i)$$

FIGURE 1.   A sign-graded poset, its two labelings and the corresponding rank function.
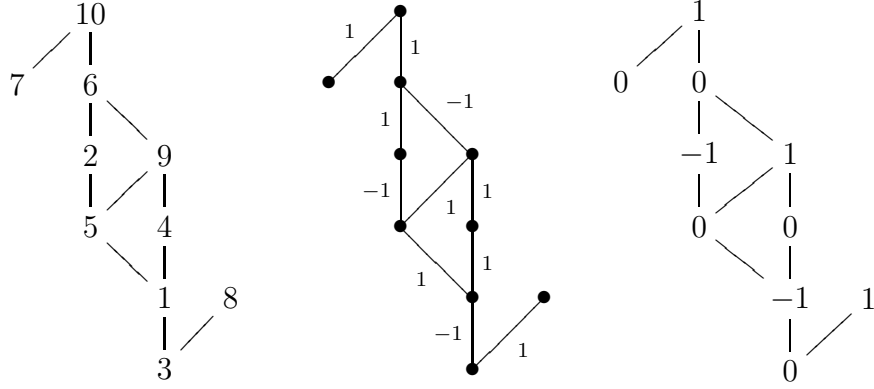
is the same. The common value of the above sum is called the *rank* of (P, ω) and is denoted $r(\epsilon)$.

We say that the poset P is ϵ-*consistent* if for every $y \in P$ the principal order ideal $\Lambda_y = \{x \in P : x \le y\}$ is ϵ_y-graded, where ϵ_y is ϵ restricted to $E(\Lambda_y)$. The *rank function* $\rho : P \to \mathbb{Z}$ of an ϵ-consistent poset P is defined by $\rho(x) = r(\epsilon_x)$. Hence, an ϵ-consistent poset P is ϵ-graded if and only if ρ is constant on the set of maximal elements.

See Fig. 1 for an example of a sign-graded poset. Note that if ϵ is identically equal to 1, i.e., if (P, ω) is naturally labeled, then a sign-graded poset with respect to ϵ is just a graded poset. Note also that if P is ϵ-graded then P is also $-\epsilon$-graded, where $-\epsilon$ is defined by $(-\epsilon)(x, y) = -\epsilon(x, y)$. Up to a shift, the order polynomial of a sign-graded labeled poset only depends on the underlying poset:

Theorem 2.2. *Let P be ϵ-graded and μ-graded. Then*

$$\Omega(P, \epsilon; t - \frac{r(\epsilon)}{2}) = \Omega(P, \mu; t - \frac{r(\mu)}{2}).$$

Proof. Let ρ_ϵ and ρ_μ denote the rank functions of (P, ϵ) and (P, μ) respectively, and let $\mathcal{A}(\epsilon)$ denote the set of (P, ϵ)-partitions. Define a function $\xi : \mathcal{A}(\epsilon) \to \mathbb{Q}^P$ by $\xi\sigma(x) = \sigma(x) + \Delta(x)$, where

$$\Delta(x) = \frac{r(\epsilon) - \rho_\epsilon(x)}{2} - \frac{r(\mu) - \rho_\mu(x)}{2}.$$

The four possible combinations of labelings of a covering-relation $(x, y) \in E$ are given in Table 1.

According to the table $\xi\sigma$ is a (P, μ)-partition provided that $\xi\sigma(x) > 0$ for all $x \in P$. But $\xi\sigma$ is order-reversing so it attains its minima on maximal elements and if z is a maximal element we have $\xi\sigma(z) = \sigma(z)$. Hence $\xi :$

TABLE 1.

$\epsilon(x, y)$	$\mu(x, y)$	σ	Δ	$\xi\sigma$
1	1	$\sigma(x) \geq \sigma(y)$	$\Delta(x) = \Delta(y)$	$\xi\sigma(x) \geq \xi\sigma(y)$
1	-1	$\sigma(x) \geq \sigma(y)$	$\Delta(x) = \Delta(y) + 1$	$\xi\sigma(x) > \xi\sigma(y)$
-1	1	$\sigma(x) > \sigma(y)$	$\Delta(x) = \Delta(y) - 1$	$\xi\sigma(x) \geq \xi\sigma(y)$
-1	-1	$\sigma(x) > \sigma(y)$	$\Delta(x) = \Delta(y)$	$\xi\sigma(x) > \xi\sigma(y)$

$\mathcal{A}(\epsilon) \to \mathcal{A}(\mu)$. By symmetry we also have a map $\eta : \mathcal{A}(\mu) \to \mathcal{A}(\epsilon)$ defined by

$$\eta\sigma(x) = \sigma(x) + \frac{r(\mu) - \rho_\mu(x)}{2} - \frac{r(\epsilon) - \rho_\epsilon(x)}{2}.$$

Hence, $\eta = \xi^{-1}$ and ξ is a bijection.

Since σ and $\xi\sigma$ are order-reversing they attain their maxima on minimal elements. But if z is a minimal element then $\xi\sigma(z) = \sigma(z) + \frac{r(\epsilon) - r(\mu)}{2}$, which gives

$$\Omega(P, \mu; n) = \Omega(P, \epsilon; n + \frac{r(\mu) - r(\epsilon)}{2}),$$

for all non-negative integers n and the theorem follows. $\qquad\square$

Theorem 2.3. *Let P be ϵ-graded. Then*

$$\Omega(P, \epsilon; t) = (-1)^p \Omega(P, \epsilon; -t - r(\epsilon)).$$

Proof. We have the following reciprocity for order polynomials, see [12]:

$$\Omega(P, -\epsilon; t) = (-1)^p \Omega(P, \epsilon; -t). \tag{2.1}$$

Note that $r(-\epsilon) = -r(\epsilon)$, so by Theorem 2.2 we have:

$$\Omega(P, -\epsilon; t) = \Omega(P, \epsilon, t - r(\epsilon)),$$

which, combined with (2.1), gives the desired result. $\qquad\square$

Corollary 2.4. *Let P be an ϵ-graded poset. Then $W(P, \epsilon, t)$ is symmetric with center of symmetry $(p - r(\epsilon) - 1)/2$. If P is also μ-graded then*

$$W(P, \mu; t) = t^{(r(\epsilon) - r(\mu))/2} W(P, \epsilon; t).$$

Proof. It is known, see [12], that if $W(P, \epsilon; t) = \sum_{i \geq 0} w_i(P, \epsilon) t^i$ then $\Omega(P, \epsilon; t) = \sum_{i \geq 0} w_i(P, \epsilon) \binom{t+p-1-i}{p}$. Let $r = r(\epsilon)$. Theorem 2.3 gives:

$$\begin{aligned}
\Omega(P, \epsilon; t) &= \sum_{i \geq 0} w_i(P, \epsilon)(-1)^p \binom{-t - r + p - 1 - i}{p} \\
&= \sum_{i \geq 0} w_i(P, \epsilon) \binom{t + r + i}{p} \\
&= \sum_{i \geq 0} w_{p-r-1-i}(P, \epsilon) \binom{t + p - 1 - i}{p},
\end{aligned}$$

so $w_i(P, \epsilon) = w_{p-r-1-i}(P, \epsilon)$ for all i, and the symmetry follows. The relationship between the W-polynomials of ϵ and μ follows from Theorem 2.2 and the expansion of order-polynomials in the basis $\binom{t+p-1-i}{p}$. $\square$

We say that a poset P is *parity graded* if the size of all maximal chains in P have the same parity. Also, a poset is P is *parity consistent* if for all $x \in P$ the order ideal Λ_x is parity graded. These classes of posets were studied in [11] in a different context. The following theorem tells us that the class of sign-graded posets is considerably greater than the class of graded posets.

Theorem 2.5. *Let P be a poset. Then*

- *there exists a labeling $\epsilon : E \to \{-1, 1\}$ such that P is ϵ-consistent if and only if P is parity consistent,*
- *there exists a labeling $\epsilon : E \to \{-1, 1\}$ such that P is ϵ-graded if and only if P is parity graded.*

Moreover, the labeling ϵ can be chosen so that the corresponding rank function has values in $\{0, 1\}$.

Proof. It suffices to prove the equivalence regarding parity graded posets. It is clear that if P is ϵ-graded then P is parity graded.

Let P be parity graded. Then, for any $x \in P$, all saturated from a minimal element to x has the same length modulo 2. Hence, we may define a labeling $\epsilon : P \to \{-1, 1\}$ by $\epsilon(x, y) = (-1)^{\ell(x)}$, where $\ell(x)$ is the length of any saturated chain starting at a minimal element and ending at x. It follows that P is ϵ-graded and that its rank function has values in $\{0, 1\}$. $\square$

We say that $\omega : P \to \{1, 2, \ldots, p\}$ is *canonical* if (P, ω) has a rank-function ρ with values in $\{0, 1\}$, and $\rho(x) < \rho(y)$ implies $\omega(x) < \omega(y)$. By Theorem 2.5 we know that P admits a canonical labeling if P is ϵ-consistent for some ϵ.

3. The Jordan-Hölder set of an ϵ-consistent poset

Let P be ω-consistent. We may assume that $\omega(x) < \omega(y)$ whenever $\rho(x) < \rho(y)$. Suppose that $x, y \in P$ are incomparable and that $\rho(y) = \rho(x)+1$. Then the Jordan-Hölder set of (P, ω) can be partitioned into two sets: One where in all permutations $\omega(x)$ comes before $\omega(y)$ and one where $\omega(y)$ always comes before $\omega(x)$. This means that $\mathcal{L}(P, \omega)$ is the disjoint union

$$\mathcal{L}(P, \omega) = \mathcal{L}(P', \omega) \sqcup \mathcal{L}(P'', \omega), \tag{3.1}$$

where P' is the transitive closure of $E \cup \{x \prec y\}$, and P'' is the transitive closure of $E \cup \{y \prec x\}$.

Lemma 3.1. *With definitions as above P' and P'' are ω-consistent with the same rank-function as (P, ω).*

Proof. Let $c : z_0 \prec z_1 \prec \cdots \prec z_k = z$ be a saturated chain in P'', where z_0 is a minimal element of P''. Of course z_0 is also a minimal element of P. We have to prove that

$$\rho(z) = \sum_{i=0}^{k-1} \epsilon''(z_i, z_{i+1}),$$

where ϵ'' is the labeling of $E(P'')$ and ρ is the rank-function of (P, ω).

All covering relations in P'', except $y \prec x$, are also covering relations in P. If y and x do not appear in c, then c is a saturated chain in P and there is nothing to prove. Otherwise

$$c : y_0 \prec \cdots \prec y_i = y \prec x = x_{i+1} \prec x_{i+2} \prec \cdots \prec x_k = z.$$

Note that if $s_0 \prec s_1 \prec \cdots \prec s_\ell$ is any saturated chain in P then $\sum_{i=0}^{\ell-1} \epsilon(s_i, s_{i+1}) = \rho(s_\ell) - \rho(s_0)$. Since $y_0 \prec \cdots \prec y_i = y$ and $x = x_{i+1} \prec x_{i+2} \prec \cdots \prec x_k = z$ are saturated chains in P we have

$$
\begin{aligned}
\sum_{i=0}^{k-1} \epsilon''(z_i, z_{i+1}) &= \rho(y) + \epsilon''(y, x) + \rho(z) - \rho(x) \\
&= \rho(y) - 1 - \rho(x) + \rho(z) \\
&= \rho(z),
\end{aligned}
$$

as was to be proved. The statement for (P', ω) follows similarly. $\square$

We say that a ω-consistent poset P is *saturated* if for all $x, y \in P$ we have that x and y are comparable whenever $|\rho(y) - \rho(x)| = 1$. Let P and Q be posets on the same set. Then Q *extends* P if $x <_Q y$ whenever $x <_P y$.

Theorem 3.2. *Let P be a ω-consistent poset. Then the Jordan-Hölder set of (P, ω) is uniquely decomposed as the disjoint union*

$$\mathcal{L}(P, \omega) = \bigsqcup_Q \mathcal{L}(Q, \omega),$$

where the union is over all saturated ω-consistent posets Q that extend P and have the same rank-function as (P, ω).

Proof. That the union exhausts $\mathcal{L}(P, \omega)$ follows from (3.1) and Lemma 3.1. Let Q_1 and Q_2 be two different saturated ω-consistent posets that extend P and have the same rank-function as (P, ω). We may assume that Q_2 does not extend Q_1. Then there exists a covering relation $x \prec y$ in Q_1 such that $x \not< y$ in Q_2. Since $|\rho(x) - \rho(y)| = 1$ we must have $y < x$ in Q_2. Thus $\omega(x)$ precedes $\omega(y)$ in any permutation in $\mathcal{L}(Q_1, \omega)$, and $\omega(y)$ precedes $\omega(x)$ in any permutation in $\mathcal{L}(Q_2, \omega)$. Hence, the union is disjoint and unique. $\square$

We need two operations on labeled posets: Let (P, ϵ) and (Q, μ) be two labeled posets. The *ordinal sum*, $P \oplus Q$, of P and Q is the poset with the disjoint union of P and Q as underlying set and with partial order defined

by $x \leq y$ if $x \leq_P y$ or $x \leq_Q y$, or $x \in P, y \in Q$. Define two labelings of $E(P \oplus Q)$ by

$$
\begin{aligned}
(\epsilon \oplus_1 \mu)(x,y) &= \epsilon(x,y) \text{ if } (x,y) \in E(P), \\
(\epsilon \oplus_1 \mu)(x,y) &= \mu(x,y) \text{ if } (x,y) \in E(Q) \text{ and} \\
(\epsilon \oplus_1 \mu)(x,y) &= 1 \text{ otherwise.} \\
(\epsilon \oplus_{-1} \mu)(x,y) &= \epsilon(x,y) \text{ if } (x,y) \in E(P), \\
(\epsilon \oplus_{-1} \mu)(x,y) &= \mu(x,y) \text{ if } (x,y) \in E(Q) \text{ and} \\
(\epsilon \oplus_{-1} \mu)(x,y) &= -1 \text{ otherwise.}
\end{aligned}
$$

With a slight abuse of notation we write $P \oplus_{\pm 1} Q$ when the labelings of P and Q are understood from the context. Note that ordinal sums are associative, i.e., $(P \oplus_{\pm 1} Q) \oplus_{\pm 1} R = P \oplus_{\pm 1} (Q \oplus_{\pm 1} R)$, and preserve the property of being sign-graded. The following result is easily obtained by combinatorial reasoning, see [2, 16]:

Proposition 3.3. *Let (P, ω) and (Q, ν) be two labeled posets. Then*

$$
W(P \oplus Q, \omega \oplus_1 \nu; t) = W(P, \omega; t)W(Q, \nu; t)
$$

and

$$
W(P \oplus Q, \omega \oplus_{-1} \nu; t) = tW(P, \omega; t)W(Q, \nu; t).
$$

Proposition 3.4. *Suppose that (P, ω) is a saturated canonically labeled ω-consistent poset. Then (P, ω) is the direct sum*

$$
(P, \omega) = A_0 \oplus_1 A_1 \oplus_{-1} A_2 \oplus_1 A_3 \oplus_{-1} \cdots \oplus_{\pm 1} A_k,
$$

where the A_is are anti-chains.

Proof. Let $\pi \in \mathcal{L}(P, \omega)$. Then we may write π as $\pi = w_0 w_1 \cdots w_k$ where the w_is are maximal words with respect to the property: If a and b are letters of w_i then $\rho(\omega^{-1}(a)) = \rho(\omega^{-1}(b))$. Hence $\pi \in J(Q, \omega)$ where

$$
(Q, \omega) = A_0 \oplus_1 A_1 \oplus_{-1} A_2 \oplus_1 A_3 \oplus_{-1} \cdots \oplus_{\pm 1} A_k,
$$

and A_i is the anti-chain consisting of the elements $\omega^{-1}(a)$, where a is a letter of w_i (A_i is an anti-chain, since if $x < y$ where $x, y \in A_i$ there would be a letter in π between $\omega(x)$ and $\omega(y)$ whose rank was different than that of x, y). Now, (Q, ω) is saturated so $P = Q$. $\square$

Note that the argument in the above proof also can be used to give a simple proof of Theorem 3.2 when ω is canonical.

4. The W-polynomial of a sign-graded poset

The space S^d of symmetric polynomials in $\mathbb{R}[t]$ with center of symmetry $d/2$ has a basis

$$
B_d = \{t^i(1+t)^{d-2i}\}_{i=0}^{\lfloor d/2 \rfloor}.
$$

If $h \in S^d$ has non-negative coefficients in this basis it follows immediately that the coefficients of h in the standard basis are unimodal. Let S_+^d be the

non-negative span of B_d. Thus S_+^d is a cone. Another property of S_+^d is that if $h \in S_+^d$ then it has the correct sign at -1 i.e.,

$$(-1)^{d/2} h(-1) \geq 0.$$

Lemma 4.1. *Let $c, d \in \mathbb{N}$. Then*

$$S^c S^d \subset S^{c+d}$$
$$S_+^c S_+^d \subset S_+^{c+d}.$$

Suppose further that $h \in S^d$ has positive leading coefficient and that all zeros of h are real and non-positive. Then $h \in S_+^d$.

Proof. The inclusions are obvious. Since $t \in S_+^2$ and $(1 + t) \in S_+^1$ we may assume that none of them divides h. But then we may collect the zeros of h in pairs $\{\theta, \theta^{-1}\}$. Let $A_\theta = -\theta - \theta^{-1}$. Then

$$h = C \prod_{\theta < -1} (t^2 + A_\theta t + 1),$$

where $C > 0$. Since $A_\theta > 2$ we have

$$t^2 + A_\theta t + 1 = (t + 1)^2 + (A_\theta - 2)t \in S_+^2,$$

and the lemma follows. $\square$

We can now prove our main theorem.

Theorem 4.2. *Suppose that (P, ω) is a sign-graded poset of rank r. Then $W(P, \omega; t) \in S_+^{p-r-1}$.*

Proof. By Corollary 2.4 and Lemma 2.5 we may assume that (P, ω) is canonically labeled. If Q extends P then the maximal elements of Q are also maximal elements of P. By Theorem 3.2 we know that

$$W(P, \omega; t) = \sum_Q W(Q, \omega; t),$$

where (Q, ω) is saturated and sign-graded with the same rank function and rank as (P, ω). The W-polynomials of anti-chains are the Eulerian polynomials, which have real non-negative zeros only. By Propositions 3.3 and 3.4 the polynomial $W(Q, \omega; t)$ has only real non-positive zeros so by Lemma 4.1 and Corollary 2.4 we have $W(Q, \omega; t) \in S_+^{p-r-1}$. The theorem now follows since S_+^{p-r-1} is a cone. $\square$

Corollary 4.3. *Let (P, ω) be sign-graded of rank r. Then $W(P, \omega; t)$ is symmetric and its coefficients are unimodal. Moreover, $W(P, \omega; t)$ has the correct sign at -1, i.e.,*

$$(-1)^{(p-1-r)/2} W(P, \omega; -1) \geq 0.$$

Corollary 4.4. *Let P be a graded poset. Suppose that $\Delta_{eq}(P)$ is flag. Then the Charney-Davis Conjecture holds for $\Delta_{eq}(P)$.*

Theorem 4.5. *Suppose that P is an ω-consistent poset and that $|\rho(x) - \rho(y)| \leq 1$ for all maximal elements $x, y \in P$. Then $W(P, \omega; t)$ has unimodal coefficients.*

Proof. Suppose that the ranks of maximal elements are contained in $\{r, r+1\}$. If Q is any saturated poset that extends P and has the same rank function as (P, ω) then Q is ω-graded of rank r or $r + 1$. By Theorems 3.2 and 4.2 we know that

$$W(P, \omega; t) = \sum_Q W(Q, \omega; t),$$

where $W(Q, \omega; t)$ is symmetric and unimodal with center of symmetry at $(p - 1 - r)/2$ or $(p - 2 - r)/2$. The sum of such polynomials is again unimodal.   □

5. The Charney-Davis quantity

In [8] Reiner, Stanton and Welker defined the *Charney-Davis quantity* of a graded naturally labeled poset (P, ω) of rank r to be

$$CD(P, \omega) = (-1)^{(p-1-r)/2} W(P, \omega; -1).$$

We define it in the exact same way for sign-graded posets. Since the particular labeling does not matter we write $CD(P)$. Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be any permutation. We say that π is *alternating* if $\pi_1 > \pi_2 < \pi_3 > \cdots$ and *reverse alternating* if $\pi_1 < \pi_2 > \pi_3 < \cdots$. Let (P, ω) be a canonically labeled sign-graded poset. If $\pi \in \mathcal{L}(P, \omega)$ then we may write π as $\pi = w_0 w_1 \cdots w_k$ where w_i are maximal words with respect to the property: If a and b are letters of w_i then $\rho(\omega^{-1}(a)) = \rho(\omega^{-1}(b))$. The words w_i are called the *components* of π. The following theorem is well known, see for example [10], and gives the Charney-Davis quantity of an anti-chain.

Proposition 5.1. *Let $n \geq 0$ be an integer. Then $(-1)^{(n-1)/2} A_n(-1)$ is equal to 0 if n is even and equal to the number of (reverse) alternating permutations of the set $\{1, 2, \ldots, n\}$ if n is odd.*

Theorem 5.2. *Let (P, ω) be a canonically labeled sign-graded poset. Then the Charney-Davis quantity, $CD(P)$, is equal to the number of reverse alternating permutations in $\mathcal{L}(P, \omega)$ such that all components have an odd numbers of letters.*

Proof. It suffices to prove the theorem when (P, ω) is saturated. By Proposition 3.4 we know that

$$(P, \omega) = A_0 \oplus_1 A_1 \oplus_{-1} A_2 \oplus_1 A_3 \oplus_{-1} \cdots \oplus_{\pm 1} A_k,$$

where the A_is are anti-chains. Thus $CD(P) = CD(A_0)CD(A_1) \cdots CD(A_k)$. Let $\pi = w_0 w_1 \cdots w_k \in \mathcal{L}(P, \omega)$ where w_i is a permutation of $\omega(A_i)$. Then π is a reverse alternating such that all components have an odd numbers of letters if and only if, for all i, w_i is reverse alternating if i is even and alternating if i is odd. Hence, by Proposition 5.1, the number of such permutations is indeed $CD(A_0)CD(A_1) \cdots CD(A_k)$.   □

If $h(t)$ is any polynomial with integer coefficients and $h(t) \in S^d$, it follows that $h(t)$ has integer coefficients in the basis $t^i(1+t)^{d-2i}$. Thus we know that if (P, ω) is sign-graded of rank r, then

$$W(P, \omega; t) = \sum_{i=0}^{\lfloor (p-r-1)/2 \rfloor} a_i(P, \omega) t^i (1+t)^{p-r-1-2i},$$

where $a_i(P, \omega)$ are non-negative integers. By Theorem 5.2 we have a combinatorial interpretation of the $a_{(p-r-1)/2}(P, \omega)$. A similar but more complicated interpretation of $a_i(P, \omega)$, $i = 0, 1, \ldots, \lfloor (p-r-1)/2 \rfloor$ can be deduced from Proposition 3.4 and the work in [10]. We omit this.

6. THE RIGHT MODE

Let $f(x) = a_0 + a_1 x + \cdots + a_d x^d$ be a polyomial with real coefficients. The *mode*, $\mathrm{mode}(f)$, of f is the average value of the indices i such that $a_i = \max\{a_j\}_{j=0}^d$. One can easily compute the mode of a polynomial with real non-positive zeros only:

Theorem 6.1. [4] *Let f be a polynomial with real non-positive zeros only and with positive leading coefficient. Then*

$$\left| \frac{f'(1)}{f(1)} - \mathrm{mode}(f) \right| < 1.$$

It is known, see [2, 12, 16], that

$$W(P, \omega; x) = \sum_{i=1}^{p} e_i(P, \omega) x^{i-1} (1-x)^{p-i},$$

where $e_i(P, \omega)$ is the number of surjective (P, ω)-partitions $\sigma : P \to \{1, 2, \ldots, i\}$. A simple calculation gives

$$\frac{W'(P, \omega; 1)}{W(P, \omega; 1)} = p - 1 - \frac{e_{p-1}(P, \omega)}{e_p(P, \omega)}. \tag{6.1}$$

If P is ω-graded of rank r we know by Theorem 4.2 that $\mathrm{mode}(W(P, \omega; x)) = (p-r-1)/2$. The Neggers-Stanley conjecture, Theorem 6.1 and (6.1) suggest that $2e_{p-1}(P, \omega) = (p + r - 1)e_p(P, \omega)$. Stanley [12] proved this for graded posets and it generalizes to sign-graded posets:

Proposition 6.2. *Let P be ω-graded of rank r. Then*

$$2e_{p-1}(P, \omega) = (p + r - 1)e_p(P, \omega).$$

Proof. The identity follows when expanding $\Omega(P, \omega, t)$ in powers of t using Theorem 2.3. See [12, Corollary 19.4] for details. $\square$

7. A characterization of sign-graded posets

Here we give a characterization of sign-graded posets along the lines of the characterization of graded posets given by Stanley in [12]. Let (P, ϵ) be a labeled poset. Define a function $\delta = \delta_\epsilon : P \to \mathbb{Z}$ by

$$\delta(x) = \max\{\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i)\},$$

where $x = x_0 \prec x_1 \prec \cdots \prec x_\ell$ is any saturated chain starting at x and ending at a maximal element x_ℓ. Define a map $\Phi = \Phi_\epsilon : \mathcal{A}(\epsilon) \to \mathbb{Z}^P$ by

$$\Phi\sigma = \sigma + \delta.$$

We have

$$\delta(x) \geq \delta(y) + \epsilon(x, y). \tag{7.1}$$

This means that $\Phi\sigma(x) > \Phi\sigma(y)$ if $\epsilon(x, y) = 1$ and $\Phi\sigma(x) \geq \Phi\sigma(y)$ if $\epsilon(x, y) = -1$. Thus $\Phi\sigma$ is a $(P, -\epsilon)$-partition provided that $\Phi\sigma(x) > 0$ for all $x \in P$. But $\Phi\sigma$ is order reversing so it attains its minimum at maximal elements and for maximal elements, z, we have $\Phi\sigma(z) = \sigma(z)$. This shows that $\Phi : \mathcal{A}(\epsilon) \to \mathcal{A}(-\epsilon)$ is an injection.

The *dual*, (P^*, ϵ^*), of a labeled poset (P, ϵ) is defined by $x <_{P^*} y$ if and only if $y <_{P^*} x$, with labeling defined by $\epsilon^*(y, x) = -\epsilon(x, y)$. We say that P is *dual ϵ-consistent* if P^* is ϵ^*-consistent.

Proposition 7.1. *Let (P, ϵ) be labeled poset. Then $\Phi_\epsilon : \mathcal{A}(\epsilon) \to \mathcal{A}(-\epsilon)$ is a bijection if and only if P is dual ϵ-consistent.*

Proof. If P is dual ϵ-consistent then P is also dual $-\epsilon$-consistent and $\delta_{-\epsilon}(x) = -\delta_\epsilon(x)$ for all $x \in P$. Thus the if part follows since the inverse of Φ_ϵ is $\Phi_{-\epsilon}$.

For the only if direction note that P is dual ϵ-consistent if and only if for all $(x, y) \in E$ we have

$$\delta(x) = \delta(y) + \epsilon(x, y)$$

Hence, if P is not dual ϵ-consistent then by (7.1), there is a covering relation $(x_0, y_0) \in E$ such that either $\epsilon(x_0, y_0) = 1$ and $\delta(x_0) \geq \delta(y_0) + 2$ or $\epsilon(x_0, y_0) = -1$ and $\delta(x_0) \geq \delta(y_0)$.

Suppose that $\epsilon(x_0, y_0) = 1$. It is clear that there is a $\sigma \in \mathcal{A}(-\epsilon)$ such that $\sigma(x_0) = \sigma(y_0) + 1$. But then

$$\sigma(x_0) - \delta(x_0) \leq \sigma(y_0) - \delta(y_0) - 1,$$

so $\sigma - \delta \notin \mathcal{A}(\epsilon)$.

Similarly, if $\epsilon(x_0, y_0) = -1$ then we can find a partition $\sigma \in \mathcal{A}(-\epsilon)$ with $\sigma(x_0) = \sigma(y_0)$, and then

$$\sigma(x_0) - \delta(x_0) \leq \sigma(y_0) - \delta(y_0),$$

so $\sigma - \delta \notin \mathcal{A}(\epsilon)$. $\qquad\square$

Let (P, ϵ) be a labeled poset. Define $r(\epsilon)$ by

$$r(\epsilon) = \max\{\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i) : x_0 \prec x_1 \prec \cdots \prec x_\ell \text{ is maximal}\}.$$

We then have:

$$\begin{aligned}
\max\{\Phi\sigma(x) : x \in P\} &= \max\{\sigma(x) + \delta_\epsilon(x) : x \text{ is minimal}\} \\
&\leq \max\{\sigma(x) : x \in P\} + r(\epsilon).
\end{aligned}$$

So if we let $\mathcal{A}_n(\epsilon)$ be the (P, ϵ)-partitions with largest part at most n we have that $\Phi_\epsilon : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ is an injection. A labeling ϵ of P is said to satisfy the λ-*chain condition* if for every $x \in P$ there is a maximal chain $c : x_0 \prec x_1 \prec \cdots \prec x_\ell$ containing x such that $\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i) = r(\epsilon)$.

Lemma 7.2. *Suppose that n is a non-negative integer such that $\Omega(P, \epsilon; n) \neq 0$. If*

$$\Omega(P, -\epsilon; n + r(\epsilon)) = \Omega(P, \epsilon; n)$$

then ϵ satisfies the λ-chain condition.

Proof. Define $\delta^* : P \to \mathbb{Z}$ by

$$\delta^*(x) = \max\{\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i) : x_0 \prec x_1 \prec \cdots \prec x_\ell = x\},$$

where the maximum is taken over all maximal chains starting at a minimal element and ending at x. Then

$$\delta(x) + \delta^*(x) \leq r(\epsilon) \tag{7.2}$$

for all x, and ϵ satisfies the λ-chain condition if and only if we have equality in (7.2) for all $x \in P$. It is easy to see that the map $\Phi^* : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ defined by

$$\Phi^*\sigma(x) = \sigma(x) + r(\epsilon) - \delta^*(x),$$

is well-defined and is an injection. By (7.2) we have $\Phi\sigma(x) \leq \Phi^*\sigma(x)$ for all σ and all $x \in P$, with equality if and only if x is in a maximal chain of maximal weight. This means that in order for $\Phi : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ to be a bijection it is necessary for ϵ to satisfy the λ-chain condition. $\square$

Theorem 7.3. *Let ϵ be a labeling of P. Then*

$$\Omega(P, \epsilon; t) = (-1)^p \Omega(P, \epsilon; -t - r(\epsilon))$$

if and only if P is ϵ-graded of rank $r(\epsilon)$.

Proof. The "if" part is Theorem 2.3, so suppose that the equality of the theorem holds. By reciprocity we have

$$(-1)^p \Omega(P, \epsilon; -t - r(\epsilon)) = \Omega(P, -\epsilon; t + r(\epsilon)),$$

and since $\Phi_\epsilon : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ is an injection it is also a bijection. By Proposition 7.1 we have that P is dual ϵ-consistent and by Lemma 7.2, we have that all minimal elements are members of maximal chains of maximal weight. In other words P is ϵ-graded. $\square$

It should be noted that it is not necessary for P to be ϵ-graded in order for $W(P, \epsilon; t)$ to be symmetric. For example, if (P, ϵ) is any labeled poset then the W-polynomial of the disjoint union of (P, ϵ) and $(P, -\epsilon)$ is easily seen to be symmetric. However, we have the following:

Corollary 7.4. *Suppose that*

$$\Omega(P, \epsilon; t) = \Omega(P, -\epsilon; t + s),$$

for some $s \in \mathbb{Z}$. Then $-r(-\epsilon) \le s \le r(\epsilon)$, with equality if and only if P is ϵ-graded.

Proof. We have an injection $\Phi_\epsilon : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$. This means that $s \le r(\epsilon)$. The lower bound follows from the injection $\Phi_{-\epsilon}$, and the statement of equality follows from Theorem 7.3. $\square$

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